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TRANSIENT THERMAL STRESS INTENSITY FACTORS FOR A CRACK IN A SEMI-INFINITE PLATE OF A FUNCTIONALLY GRADIENT MATERIAL

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Abstract—A crack in a semi-infinite plate of a functionally gradient material is studied under thermal shock loading conditions. The crack faces are supposed to be completely insulated. All material properties are assumed to be exponentially dependent on the distance from the crack line parallel to the boundary of the plate. By using both the Laplace transform and the Fourier transform, the thermal and mechanical problems are reduced to two systems of singular integral equations which are solved numerically. The stress intensity factors vs time for various material constants are calculated. The results show that by selecting the material constants appropriately, the stress intensity factors can be reduced substantially.

1. INTRODUCTION

It is well known that in aerospace and nuclear engineering, many structural components are subject to severe thermal loading which gives rise to intense thermal stresses in the components especially around cracks and other kinds of defects. The concentration of stresses around defects often results in catastrophe. In recent years, the concept of so-called functionally gradient materials (FGM) has been introduced and applied to the development of structural components (Koizumi, 1993). The advantages of FGM materials are that the material can resist high temperatures effectively and, in the meantime, thermal stresses in the material can be reduced significantly (Noda and Tsuji, 1990; Arai *et al.*, 1991). The interest in FGM research is growing rapidly due to these advantages. Usually FGM materials are mixtures of ceramics and metals fabricated in such a way that the volume fractions of the constituents are varied continuously in a predetermined composition profile. The materials thus obtained have both mechanical and thermal nonhomogeneities. Therefore, the nonhomogeneous continuum theory can offer the basis for evaluating the mechanical and thermal properties of FGM materials.

A few investigations on thermal stresses around cracks in nonhomogeneous materials or FGM have been made. Among them are Erdogan and Wu (1993), Jin and Noda (1993a) and Noda and Jin (1993), but only steady thermal loading was considered. It was found that by selecting the material constants appropriately, the steady thermal stress intensity factors can be lowered substantially (Jin and Noda, 1993a; Noda and Jin, 1993).

In this paper, a crack in a semi-infinite plate of a functionally gradient material mathematically modeled by a nonhomogeneous solid is studied under transient thermal loading conditions. It is assumed that initially the medium is at the uniform temperature zero and is suddenly subjected to a uniform temperature T_0 along the traction-free boundary. The crack faces (parallel to the boundary) are supposed to remain completely insulated. We assume that all material properties depend only on the coordinate y (perpendicular to the crack faces) in such a way that the properties are some exponential functions of y. By using both the Laplace transform and the Fourier transform, the thermal and mechanical problems are reduced to two systems of singular integral equations. The equations are solved numerically and the stress intensity factors vs time for various material constants are calculated.

2. BASIC EQUATIONS

As shown in Fig. 1, consider a semi-infinite plate containing a through crack with its length being $2c$ and denote by (x, y) the rectangular coordinate system with its origin at the middle point of the crack face and x direction along with the crack line. It is assumed that initially the medium is at the uniform temperature zero and is suddenly subjected to a uniform temperature T_0 along the traction-free boundary $y = -a$. The crack faces are assumed to remain completely insulated.

The basic equations of nonhomogeneous solids expressed by the Airy stress function F and the temperature T are (Jin and Noda, 1993b)

$$\nabla^2\left(\frac{1}{E}\nabla^2 F\right) - \frac{\partial^2}{\partial y^2}\left(\frac{1+\nu}{E}\right)\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left(\frac{1+\nu}{E}\right)\frac{\partial^2 F}{\partial y^2} + 2\frac{\partial^2}{\partial x \partial y}\left(\frac{1+\nu}{E}\right)\frac{\partial^2 F}{\partial x \partial y} + \nabla^2(\alpha T) = 0, \tag{1}$$

$$\nabla^2 T + \frac{1}{k}\nabla k \cdot \nabla T = \frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{2}$$

where E and ν are the Young's modulus and the Poisson's ratio, respectively; α, k and κ are the coefficient of linear thermal expansion, the heat conductivity and the thermal diffusivity, respectively. In eqns (1) and (2), ∇^2 is the two dimensional Laplacian operator and ∇ is the gradient operator, respectively. In this study, we assume that the thermoelastic coupling effect and the inertia effect are negligible and the problem thus considered is uncoupled and quasi-static. The previous studies on the dynamic coupling thermoelastic problems (Sternberg and Chakravorty, 1959; Noda *et al.*, 1990) seem to support the above assumption.

FGM materials usually are mixtures of ceramics (which have poorer heat conductivity and lower thermal expansion) and metals (which have higher toughness and better heat conductivity) for resisting high temperatures and reducing thermal stresses. Hence, from the thermal loading conditions studied in this paper, it is reasonable to suppose that the material possesses the following nonhomogeneous properties:

$$E = E_0 \exp(\beta y), \qquad \nu = \nu_0(1+\varepsilon y)\exp(\beta y)$$
$$\alpha = \alpha_0 \exp(\gamma y), \qquad k = k_0 \exp(\delta y), \qquad \kappa = \kappa_0, \tag{3}$$

where E_0, ν_0, α_0, k_0, κ_0 and β, ε, γ, δ are material constants. The function $\nu(y)$ given in (3) is subject to the restriction that $0 \leqslant \nu(y) \leqslant 0.5$ for the region of y considered and using this form of $\nu(y)$ was justified by Delale and Erdogan (1988) in that the Poisson's ratio does not significantly influence the stress intensity factors. In this study, it is assumed that β takes negative values since in FGM materials the ceramic exposed to thermal shock usually has higher elastic modulus than that of the metal. The condition is also consistent with the requirement that $\nu \leqslant 0.5$. It is also assumed that the thermal diffusivity κ is a constant. For some materials, κ indeed doesn't vary dramatically. The mechanical nonhomogeneities in (3) were used by Delale and Erdogan (1988) and the spatial variation of E in (3) was employed in fracture problems by a number of investigators, for example, Atkinson and List (1978), Dhaliwal and Singh (1978), Delale and Erdogan (1983) and Herrmann and

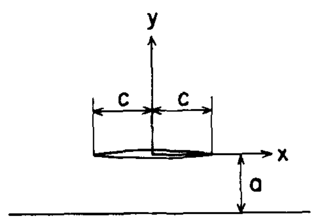

Fig. 1. Crack geometry and coordinates.

Schovanec (1990). Here we assume an exponential variation (3) of the material properties for the entire half-plane because it is both easier for mathematical treatment and practically reasonable if the crack is located not far from the boundary of the half-plane (i.e. the ratio c/a is not large).

By substituting (3) into (1) and (2) and referring to the following dimensionless variables:

$$\begin{aligned} \hat{\sigma}_{\alpha\beta} &= \sigma_{\alpha\beta}/(E_0\alpha_0 T_0), \qquad \hat{F} = F/(E_0\alpha_0 T_0 c^2) \\ (\hat{u},\hat{v}) &= (u,v)/(\alpha_0 T_0 c), \qquad \hat{\varepsilon}_{\alpha\beta} = \varepsilon_{\alpha\beta}/(\alpha_0 T_0) \\ \hat{T} &= T/T_0, \qquad \hat{t} = t/(c^2/\kappa_0) \\ (\hat{x},\hat{y},\hat{a}) &= (x,y,a)/c, \qquad (\hat{\beta},\hat{\varepsilon},\hat{\gamma},\hat{\delta}) = (\beta,\varepsilon,\gamma,\delta)\cdot c \end{aligned} \tag{4}$$

then the governing equations (1) and (2) reduce to equations with constant coefficients and have the following dimensionless forms:

$$\nabla^2\nabla^2 F - 2\beta\frac{\partial}{\partial y}(\nabla^2 F) + \beta^2\frac{\partial^2 F}{\partial y^2} + e^{(\beta+\gamma)y}\left(\nabla^2 T + 2\gamma\frac{\partial T}{\partial y} + \gamma^2 T\right) = 0, \tag{5}$$

$$\nabla^2 T + \delta\frac{\partial T}{\partial y} = \frac{\partial T}{\partial t}. \tag{6}$$

Here and in the following, the hat ^ of the dimensionless variables is omitted for simplicity. It is clear that the constant ε in (3) is not included in the above basic equations. Hence, ε will not appear in the expression of the Airy function F and stresses.

The dimensionless constitutive relations are:

$$\begin{aligned} \frac{\partial u}{\partial x} &= [-\nu_0(1+\varepsilon y)\sigma_y + \exp(-\beta y)\sigma_x] + \exp(\gamma y)T \\ \frac{\partial v}{\partial y} &= [-\nu_0(1+\varepsilon y)\sigma_x + \exp(-\beta y)\sigma_y] + \exp(\gamma y)T \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= 2[\nu_0(1+\varepsilon y) + \exp(-\beta y)]\sigma_{xy}. \end{aligned} \tag{7}$$

The initial and the boundary conditions in dimensionless forms are:

$$T = 0, \qquad t = 0, \tag{8}$$

and

$$\begin{aligned} T &= 1, \qquad y = -a, \qquad |x| < \infty, \qquad t > 0 \\ \frac{\partial T}{\partial y} &= 0, \qquad y = 0, \qquad |x| \leqslant 1 \end{aligned} \tag{9a}$$

$$\begin{aligned} T(x,0^+) &= T(x,0^-), \qquad |x| > 1 \\ \frac{\partial T(x,0^+)}{\partial y} &= \frac{\partial T(x,0^-)}{\partial y}, \qquad |x| > 1 \end{aligned} \tag{9b}$$

$$T \to 0, \qquad x^2 + y^2 \to \infty \tag{10}$$

for thermal conditions, and

$$\sigma_{xy} = \sigma_y = 0, \qquad y = 0, \qquad |x| \leqslant 1$$
$$\sigma_{xy} = \sigma_y = 0, \qquad y = -a, \qquad |x| < \infty, \tag{11a}$$

$$\sigma_{\alpha\beta} \to \text{finite}, \qquad x^2+y^2 \to \infty, \tag{11b}$$

$$\sigma_{xy}(x,0^+) = \sigma_{xy}(x,0^-), \qquad |x| > 1$$
$$\sigma_y(x,0^+) = \sigma_y(x,0^-), \qquad |x| > 1$$
$$u(x,0^+) = u(x,0^-), \qquad |x| > 1$$
$$v(x,0^+) = v(x,0^-), \qquad |x| > 1, \tag{12}$$

for mechanical conditions.

Since ε will not be included in the expressions of stresses as stated above, it is clear from (7), (11) and (12) that ε will also not be included in the boundary conditions considered in this study. Therefore, ε will not affect the final results of stress intensity factors. Not taking ε just as zero is due to the consideration that the Poisson's ratio may take a reasonable value at the boundary $y = -a$.

3. TEMPERATURE FIELD

By applying the Laplace transform to (6):

$$T^*(x,y,p) = \int_0^\infty T(x,y,t)\exp(-pt)\,\mathrm{d}t$$

$$T(x,y,t) = \frac{1}{2\pi i}\int_{Br} T^*(x,y,p)\exp(pt)\,\mathrm{d}p, \tag{13}$$

where ***Br*** is the Bromwich path, an infinite line parallel to and to the right of the imaginary axis in the Laplace transform plane (p-plane), and making use of the initial condition (8), we have:

$$\nabla^2 T^* + \delta\frac{\partial T^*}{\partial y} = pT^*. \tag{14}$$

The boundary conditions in the p-plane are:

$$T^* = 1/p, \qquad y = -a, \qquad |x| < \infty$$
$$\frac{\partial T^*}{\partial y} = 0, \qquad y = 0, \qquad |x| \leqslant 1$$
$$T^*(x,0^+) = T^*(x,0^-), \qquad |x| > 1$$
$$\frac{\partial T^*(x,0^+)}{\partial y} = \frac{\partial T^*(x,0^-)}{\partial y}, \qquad |x| > 1$$
$$T^* \to 0, \qquad x^2+y^2 \to \infty. \tag{15}$$

The temperature field $T^*(x, y, p)$ in the p-plane can be expressed as:

$$T^*(x,y,p) = T_1^*(y,p) + T_2^*(x,y,p), \tag{16}$$

where $T_1^*(y, p)$ satisfies the following equation and boundary conditions,

$$\frac{d^2T_1^*}{dy^2}+\delta\frac{dT_1^*}{dy}-pT_1^*=0, \tag{17}$$

$$\begin{aligned} T_1^* &= 1/p, \qquad y=-a \\ T_1^* &\to 0, \qquad y\to\infty, \end{aligned} \tag{18}$$

whereas $T_2^*(x, y, p)$ is subject to the relations,

$$\nabla^2T_2^*+\delta\frac{\partial T_2^*}{\partial y}-pT_2^*=0, \tag{19}$$

$$\begin{aligned} &T_2^*=0, && y=-a, \quad |x|<\infty \\ &T_2^*\to 0, && x^2+y^2\to\infty \\ &\frac{\partial T_2^*}{\partial y}=-\frac{dT_1^*}{dy}, && y=0, \quad |x|\leqslant 1 \\ &T_2^*(x,0^+)=T_2^*(x,0^-), && |x|>1 \\ &\frac{\partial T_2^*(x,0^+)}{\partial y}=\frac{\partial T_2^*(x,0^-)}{\partial y}, && |x|>1. \end{aligned} \tag{20}$$

It is easy to find from (17) and (18) that:

$$T_1^*(y,p)=\frac{1}{p}\exp[-\lambda(y+a)], \tag{21}$$

where λ is given by:

$$\lambda=\frac{\delta}{2}+\sqrt{p+\delta^2/4}. \tag{22}$$

Next, by applying the Fourier transform to (19) and using the boundary conditions (20), we have:

$$T_2^*(x,y,p)=\int_{-\infty}^{\infty}D(\xi,p)\exp(-\mu_2y-ix\xi)\,d\xi, \qquad y>0, \tag{23a}$$

$$T_2^*(x,y,p)=\int_{-\infty}^{\infty}\frac{\mu_2D(\xi,p)}{\mu_1-\mu_2\exp(-2\mu a)}\{1-\exp[-2\mu(a+y)]\} \times\exp(-\mu_1y-ix\xi)\,d\xi, \qquad y<0, \tag{23b}$$

in the above expressions, $\mu_i (i = 1, 2)$ are:

$$\mu_1=\frac{\delta}{2}-\mu, \qquad \mu_2=\frac{\delta}{2}+\mu, \qquad \mu=\sqrt{p+\xi^2+\delta^2/4}, \tag{24}$$

and $D(\xi, p)$ is expressed by the density function $\varphi^*(x, p)$ defined by:

$$\varphi^*(x,p)=\frac{\partial T^*(x,0^+,p)}{\partial x}-\frac{\partial T^*(x,0^-,p)}{\partial x}=\frac{\partial T_2^*(x,0^+,p)}{\partial x}-\frac{\partial T_2^*(x,0^-,p)}{\partial x}, \tag{25}$$

as follows

$$D(\xi,p) = -\frac{i[\mu_1-\mu_2\exp(-2\mu a)]}{4\pi\xi\mu}\int_{-1}^{1}\varphi^*(\tau,p)\exp(i\xi\tau)\,d\tau. \tag{26}$$

The density function $\varphi^*(x, p)$ satisfies the following singular integral equation:

$$\int_{-1}^{1}\left\{\frac{1}{\tau-x}+k^*(x,\tau,p)\right\}\varphi^*(\tau,p)\,d\tau = \frac{2\pi\lambda}{p}\exp(-\lambda a), \qquad |x| \leqslant 1 \tag{27}$$

and the condition:

$$\int_{-1}^{1}\varphi^*(x,p)\,dx = 0. \tag{28}$$

In eqn (27), the Fredholm kernel $k^*(x, \tau, p)$ is given by:

$$k^*(x,\tau,p) = \int_0^{\infty}\left\{1+\frac{\mu_2[\mu_1-\mu_2\exp(-2\mu a)]}{\xi\mu}\right\}\sin[(x-\tau)\xi]\,d\xi. \tag{29}$$

In the singular integral equation theory, eqn (27) under the condition (28) has the following form of the solution (Erdogan *et al.*, 1973):

$$\varphi^*(x,p) = \frac{\Phi^*(x,p)}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1, \tag{30}$$

where $\Phi^*(x, p)$ is a bounded and continuous function on the interval $[-1, 1]$ with p being a parameter. Once $\varphi^*(x, p)$ is obtained, the temperature field $T^*(x, y, p)$ in the Laplace transform domain can be easily calculated. The temperature $T(x, y, t)$ in the time domain can be obtained from $T^*(x, y, p)$ by making inverse Laplace transform. This will be discussed later in Section 5.

4. THERMAL STRESSES

We first consider the problem in the Laplace transform plane. In the p-plane, the basic equation (5) becomes:

$$\nabla^2\nabla^2F^*-2\beta\frac{\partial}{\partial y}(\nabla^2F^*)+\beta^2\frac{\partial^2F^*}{\partial y^2} = -e^{(\beta+\gamma)y}\left(\nabla^2T^*+2\gamma\frac{\partial T^*}{\partial y}+\gamma^2T^*\right), \tag{31}$$

and the boundary conditions are the same as (11) and (12) with the understanding that (11) and (12) are in the p-plane. The Laplace transforms of F, $\sigma_{\alpha\beta}$ and u, v are F^*, $\sigma^*_{\alpha\beta}$ and u^*, v^*, respectively.

The general solution of (31) satisfying the regular condition at infinity (11b) with the temperature given by (16), (21) and (23) can be expressed as follows:

$$F^*(x,y,p) = \int_{-\infty}^{\infty}(B_1+B_2y)\exp(-s_2y-ix\xi)\,d\xi-\int_{-\infty}^{\infty}C_{12}\exp[(\beta+\gamma-\mu_2)y-ix\xi]\,d\xi,$$

$$y > 0, \tag{32a}$$

$$F^*(x,y,p) = \int_{-\infty}^{\infty} \{(A_1+A_2y)+(A_3+A_4y)\exp(-2sy)\}\exp(-s_1y-ix\xi)\,d\xi$$

$$-\int_{-\infty}^{\infty} \{C_{21}+C_{22}\exp(-2\mu y)\}\exp[(\beta+\gamma-\mu_1)y-ix\xi]\,d\xi, \qquad y<0, \quad (32b)$$

in the above expressions, $B_i(\xi, p)(i = 1, 2)$ and $A_i(\xi, p)(i = 1, 2, 3, 4)$ are unknown functions, s_1, s_2 are:

$$s_1 = -\frac{\beta}{2}-s, \qquad s_2 = -\frac{\beta}{2}+s, \qquad s = \sqrt{\xi^2+\beta^2/4}, \tag{33}$$

and $C_{12}(\xi, p)$, $C_{21}(\xi, p)$ and $C_{22}(\xi, p)$ are known functions given by:

$$C_{12}(\xi,p) = [(\beta+\gamma-\mu_2)(\gamma-\mu_2)-\xi^2]^{-2}[\gamma^2+p-(2\gamma-\delta)\mu_2]D(\xi,p)$$

$$C_{21}(\xi,p) = [(\beta+\gamma-\mu_1)(\gamma-\mu_1)-\xi^2]^{-2}[\gamma^2+p-(2\gamma-\delta)\mu_1]\frac{\mu_2 D(\xi,p)}{\mu_1-\mu_2\exp(-2\mu a)}$$

$$C_{22}(\xi,p) = [(\beta+\gamma-\mu_2)(\gamma-\mu_2)-\xi^2]^{-2}[(2\gamma-\delta)\mu_2-\gamma^2-p]\frac{\mu_2\exp(-2\mu a)D(\xi,p)}{\mu_1-\mu_2\exp(-2\mu a)}. \tag{34}$$

The particular solution of (31) due to $T_1^*(y, p)$ is omitted since it doesn't influence the singular character of stresses near the crack tip and the stress intensity factors.

The stresses in the Laplace transform domain can be obtained from the Airy function (32) by the following well-known formulas:

$$\sigma_x^* = \frac{\partial^2 F^*}{\partial y^2}, \qquad \sigma_y^* = \frac{\partial^2 F^*}{\partial x^2}, \qquad \sigma_{xy}^* = -\frac{\partial^2 F^*}{\partial x\partial y}, \tag{35}$$

but the detailed expressions are not given here.

The constitutive relations in the Laplace transform domain are:

$$\frac{\partial u^*}{\partial x} = [-\nu_0(1+\varepsilon y)\sigma_y^*+\exp(-\beta y)\sigma_x^*]+\exp(\gamma y)T^*$$

$$\frac{\partial v^*}{\partial y} = [-\nu_0(1+\varepsilon y)\sigma_x^*+\exp(-\beta y)\sigma_y^*]+\exp(\gamma y)T^*$$

$$\frac{\partial u^*}{\partial y}+\frac{\partial v^*}{\partial x} = 2[\nu_0(1+\varepsilon y)+\exp(-\beta y)]\sigma_{xy}^*. \tag{36}$$

By substituting from (35) into (36) with F^* being given by (32), the displacements in the p-plane can be obtained.

Introduce two dislocation density functions $\psi_i^*(x, p)$ $(i = 1, 2)$ by:

$$\psi_1^*(x,p) = \frac{\partial u^*(x,0^+,p)}{\partial x}-\frac{\partial u^*(x,0^-,p)}{\partial x}$$

$$\psi_2^*(x,p) = \frac{\partial v^*(x,0^+,p)}{\partial x}-\frac{\partial v^*(x,0^-,p)}{\partial x}. \tag{37}$$

By using the boundary conditions (11) and (12), it can be shown that $\psi_1^*(x, p)$ $(i = 1, 2)$ satisfy the following singular integral equations:

$$\int_{-1}^{1}\sum_{j=1}^{2}\left[\frac{\delta_{ij}}{\tau-x}+k_{ij}(x,\tau)\right]\psi_j^*(\tau,p)\,\mathrm{d}\tau = 4\pi L_i^*(x,p), \qquad i=1,2, \qquad |x|\leqslant 1, \tag{38}$$

and the conditions

$$\int_{-1}^{1}\psi_i^*(x,p)\,\mathrm{d}x = 0, \qquad i=1,2, \tag{39}$$

in eqn (38), the Fredholm type kernels $k_{ij}(x,\tau)$ $(i,j=1,2)$ are given by:

$$k_{11}(x,\tau) = \int_0^\infty \{1-4\xi f_{11}(\xi)\}\sin[(x-\tau)\xi]\,\mathrm{d}\xi$$

$$k_{22}(x,\tau) = \int_0^\infty \{1-4\xi^2 f_{22}(\xi)\}\sin[(x-\tau)\xi]\,\mathrm{d}\xi$$

$$k_{12}(x,\tau) = \int_0^\infty -4\xi f_{12}(\xi)\cos[(x-\tau)\xi]\,\mathrm{d}\xi$$

$$k_{21}(x,\tau) = \int_0^\infty 4\xi^2 f_{21}(\xi)\cos[(x-\tau)\xi]\,\mathrm{d}\xi, \qquad -1\leqslant x, \qquad \tau\leqslant 1 \tag{40}$$

and the right-hand functions $L_i^*(x,p)$ $(i=1,2)$ are:

$$L_1^*(x,p) = 2\int_0^\infty \xi l_1^*(\xi,p)\sin(x\xi)\,\mathrm{d}\xi, \qquad |x|\leqslant 1$$

$$L_2^*(x,p) = -2\int_0^\infty \xi^2 l_2^*(\xi,p)\cos(x\xi)\,\mathrm{d}\xi \qquad |x|\leqslant 1 \tag{41}$$

$$l_1^*(\xi,p) = -\frac{h_{11}(\beta g_1+2g_2)+2sh_{12}(s_2g_1-g_2)}{8s^3}-g_3$$

$$l_2^*(\xi,p) = -\frac{h_{21}(\beta g_1+2g_2)+2sh_{22}(s_2g_1-g_2)}{8s^3}-g_4 \tag{42}$$

in the expressions (40), (41) and (42), $f_{ij}(\xi)$, $h_{ij}(\xi)$ $(i,j=1,2)$ and $g_i(\xi,p)$ $(i=1,2,3,4)$ are given in Appendix A.

The solutions of integral eqns (38) have the following forms:

$$\psi_i^*(x,p) = \frac{\Psi_i^*(x,p)}{\sqrt{1-x^2}}, \qquad i=1,2, \qquad |x|\leqslant 1 \tag{43}$$

where $\Psi_i^*(x,p)$ $(i=1,2)$ are continuous bounded functions in the interval $[-1,1]$ with p being a parameter.

The crack-tip stress field in the Laplace transform plane can be evaluated as a standard square root singular field, i.e.

$$\sigma_{\alpha\beta}^*(r,\theta,p) = \frac{1}{\sqrt{2\pi r}}\{K_{\mathrm{I}}^*(p)\tilde{\sigma}_{\alpha\beta}^{\mathrm{I}}(\theta)+K_{\mathrm{II}}^*(p)\tilde{\sigma}_{\alpha\beta}^{\mathrm{II}}(\theta)\} \tag{44}$$

where (r,θ) are the polar coordinates at the crack tip defined by:

$$x = 1 + r\cos\theta, \qquad y = r\sin\theta, \tag{45}$$

$\tilde{\sigma}^{\rm I}_{\alpha\beta}(\theta)$ and $\tilde{\sigma}^{\rm II}_{\alpha\beta}(\theta)$ are the angular distributions which can be found in any book on fracture mechanics, for example, Sih (1977), and the dimensionless stress intensity factors $K_{\rm I}^*(p)$ and $K_{\rm II}^*(p)$ in the p-plane are given by:

$$\{K_{\rm I}^*(p), K_{\rm II}^*(p)\} = -\frac{\sqrt{\pi}}{4}\{\Psi_2^*(1,p), \Psi_1^*(1,p)\} \tag{46}$$

By applying the inverse Laplace transform to (38), (39), (41), (43), (44) and (46), the integral equations and the crack tip stress field in the time domain are obtained as follows:

$$\int_{-1}^{1}\sum_{j=1}^{2}\left[\frac{\delta_{ij}}{\tau - x} + k_{ij}(x,\tau)\right]\psi_j(\tau,t)\,\mathrm{d}\tau = 4\pi L_i(x,t), \qquad i = 1,2, \qquad |x| \leqslant 1, \tag{47}$$

$$\int_{-1}^{1}\psi_i(x,t)\,\mathrm{d}x = 0, \qquad i = 1,2, \tag{48}$$

$$\psi_i(x,t) = \frac{\Psi_i(x,t)}{\sqrt{1-x^2}}, \qquad i = 1,2, \qquad |x| \leqslant 1, \tag{49}$$

and

$$\sigma_{\alpha\beta}(r,\theta,t) = \frac{1}{\sqrt{2\pi r}}\{K_{\rm I}(t)\tilde{\sigma}^{\rm I}_{\alpha\beta}(\theta) + K_{\rm II}(t)\tilde{\sigma}^{\rm II}_{\alpha\beta}(\theta)\}, \tag{50}$$

$$\{K_{\rm I}(t), K_{\rm II}(t)\} = -\frac{\sqrt{\pi}}{4}\{\Psi_2(1,t), \Psi_1(1,t)\}. \tag{51}$$

In the above equations, $\psi_i(x, t)$, $\Psi_i(x, t)$ and $L_i(x, t)$ are the inverse Laplace transforms of $\psi_i^*(x, p)$, $\Psi_i^*(x, p)$ and $L_i^*(x, p)$, respectively

$$\psi_i^*(x,p) = \int_0^\infty \psi_i(x,t)\exp(-pt)\,\mathrm{d}t, \tag{52}$$

$$\Psi_i^*(x,p) = \int_0^\infty \Psi_i(x,t)\exp(-pt)\,\mathrm{d}t, \tag{53}$$

$$L_i^*(x,p) = \int_0^\infty L_i(x,t)\exp(-pt)\,\mathrm{d}t, \tag{54}$$

and $\Psi_i(x, t)$ $(i = 1, 2)$ are continuous bounded functions in the interval $[-1, 1]$ with t being a parameter.

5. NUMERICAL RESULTS AND DISCUSSIONS

5.1. *Numerical inversion of the Laplace transform*

The temperature field $T^*(x, y, p)$ in the Laplace transform plane can be obtained from (16), (21), (23) and (26) once the integral equation (27) is solved. To get the temperature $T(x, y, t)$ in the time domain, we must make inverse Laplace transform from $T^*(x, y, p)$. In solving eqns (47) for the mechanical problem, the functions $L_i(x, t)$ in the right hand side also need to be evaluated from their Laplace transforms $L_i^*(x, p)$. It is very difficult to make analytical inversions, and therefore, the numerical inversion is practical and useful. Although there are a number of numerical methods, the one used here is due to Miller and Guy (1966) which has been widely used in fracture dynamics (Sih 1977; Fan, 1990). A brief description of the method is given in Appendix B.

5.2. *Temperature field*

We first discuss the numerical solution of the eqn (27). From symmetry, it is seen that $\varphi^*(x, p) = -\varphi^*(-x, p)$. Thus, the unknown function my be expressed as follows:

$$\varphi^*(x,p) = \frac{\Phi^*(x,p)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \sum_{n=1}^{\infty} a_n(p) T_{2n-1}(x), \qquad |x| \leqslant 1 \tag{55}$$

where $T_{2n-1}(x)$ are the Chebyshev polynomials of the first kind. From the orthogonality of $T_n(x)$ it follows that the condition (28) is satisfied with the function $\varphi^*(x, p)$ given by (55). By substituting (55) into (27) and using the relations:

$$\int_{-1}^{1} \frac{T_n(\tau)\,\mathrm{d}\tau}{(\tau-x)\sqrt{1-\tau^2}} = \pi U_{n-1}(x), \qquad |x| < 1, \qquad n \geqslant 1 \tag{56}$$

where $U_{n-1}(x)$ are the Chebyshev polynomials of the second kind, it is found that:

$$\sum_{n=1}^{\infty} a_n(p)\{\pi U_{2n-2}(x) + H_n(x,p)]\} = \frac{2\pi\lambda}{p} \exp(-\lambda a), \qquad |x| \leqslant 1, \tag{57}$$

where $H_n(x, p)$ are

$$H_n(x,p) = \int_{-1}^{1} \frac{k^*(x,\tau,p)}{\sqrt{1-\tau^2}} T_{2n-1}(\tau)\,\mathrm{d}\tau. \tag{58}$$

To solve the functional equations (57), both sides of (57) are expanded into series of the Chebyshev polynomials of the first kind. By comparing the coefficients and truncating the series at the Nth term, we obtain:

$$\sum_{n=1}^{N} \{F_{mn} + G_{mn}(p)\} a_n(p) = R_m(p), \qquad m = 1, 2, \ldots, N, \tag{59}$$

where

$$F_{mn} = \begin{cases} 1, & 1 \leqslant m \leqslant n, \\ 0, & m > n \end{cases}, \tag{60}$$

$$G_{mn}(p) = \frac{1}{\pi^2} \int_{-1}^{1} \frac{H_n(x,p)}{\sqrt{1-x^2}} T_{2m-2}(x)\,\mathrm{d}x, \tag{61}$$

$$R_1 = \frac{2\lambda}{p} \exp(-\lambda a), \qquad R_m = 0, \qquad 2 \leqslant m \leqslant N. \tag{62}$$

Once the coefficients $a_n(p)$ are obtained, the numerical solution of the integral equation (27) can be calculated by (55) and the temperature in the Laplace transform plane can be obtained from (16), (21), (23) and (26). The temperature in the time domain can be evaluated by using the numerical inversion of the Laplace transform described in Appendix B.

The temperature on the crack faces and the crack extended line ($y = 0$, $x > 1$) for

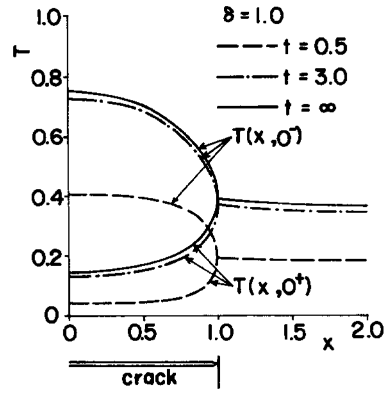

Fig. 2. The temperature on the crack faces and the crack extended line ($y = 0$) ($\delta = 1$).

different times is depicted in Figs 2 and 3. Figure 2 shows the results for $\delta = 1$ and Fig. 3 for $\delta = 2$. It is clear that the jump of the temperature across the crack faces decreases with the increase of nonhomogeneous parameter δ. Another fact is that the temperature reaches the steady state in a shorter period of time for larger value of δ than smaller δ (the dimensionless time is about 3.5 for $\delta = 1$ and 1.5 for $\delta = 2$).

5.3. *The effect of nonhomogeneity of the material on the stress intensity factors*

The stress intensity factors (SIFs) can be obtained once we get the solutions of the integral equations (47). The numerical technique for solving (47) is similar to that for the temperature problem. The final results are as follows:

$$\psi_1(x,t) = \frac{\Psi_1(x,t)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \sum_{n=1}^{\infty} b_n(t) T_{2n}(x)$$

$$\psi_2(x,t) = \frac{\Psi_2(x,t)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \sum_{n=1}^{\infty} c_n(t) T_{2n-1}(x), \tag{63}$$

$$\sum_{n=1}^{N} \{(F_{mn}+G_{mn}^{11})b_n(t)+G_{mn}^{12}c_n(t)\} = 4R_m^1(t), \qquad m = 1,2,\ldots,N$$

$$\sum_{n=1}^{N} \{G_{mn}^{21}b_n(t)+(F_{mn}+G_{mn}^{22})c_n(t)\} = 4R_m^2(t), \qquad m = 1,2,\ldots,N, \tag{64}$$

where

$$F_{mn} = \begin{cases} 1, & 1 \leqslant m \leqslant n \\ 0, & m > n \end{cases}, \tag{65}$$

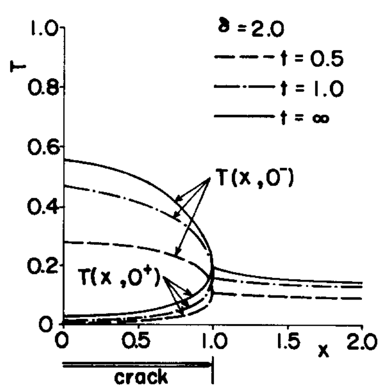

Fig. 3. The temperature on the crack faces and the crack extended line ($y = 0$) ($\delta = 2$).

$$G_{mn}^{ij} = \frac{1}{\pi^2} \int_{-1}^{1} \frac{H_n^{ij}(x)}{\sqrt{1-x^2}} T_{2m-i}(x)\,dx, \qquad i,j = 1,2, \tag{66}$$

$$H_n^{ij}(x) = \int_{-1}^{1} \frac{k_{ij}(x,\tau)}{\sqrt{1-\tau^1}} T_{2n-j+1}(\tau)\,d\tau, \qquad i,j = 1,2, \tag{67}$$

$$R_m^i(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{L_i(x,t)}{\sqrt{1-x^2}} T_{2m-i}(x)\,dx, \qquad i = 1,2. \tag{68}$$

The functions $R_m^i(t)$ in (68) can be evaluated from their Laplace transforms $R_m^{i*}(p)$

$$R_m^{i*}(p) = \frac{1}{\pi} \int_{-1}^{1} \frac{L_i^*(x,p)}{\sqrt{1-x^2}} T_{2m-i}(x)\,dx, \qquad i = 1,2, \tag{69}$$

where $L_i^*(x, p)$ are known functions of x and p given by (41).

Once the values of $b_n(t)$ and $c_n(t)$ are obtained, the stress intensity factors (SIFs) $K_I(t)$ and $K_{II}(t)$ can be calculated by the following formulae:

$$\{K_I(t), K_{II}(t)\} = -\frac{\sqrt{\pi}}{4} \sum_{n=1}^{N} \{c_n(t), b_n(t)\}. \tag{70}$$

In Figs 4–7, the SIFs K_I and K_{II} are plotted vs the dimensionless time t for $a = 1$ and

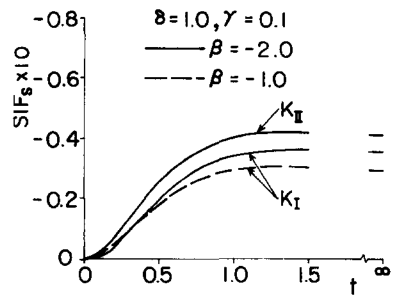

Fig. 4. The variation of SIFs with time t for different β, δ and γ.

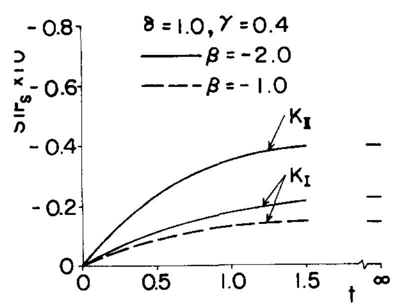

Fig. 5. The variation of SIFs with time t for different β, δ and γ.

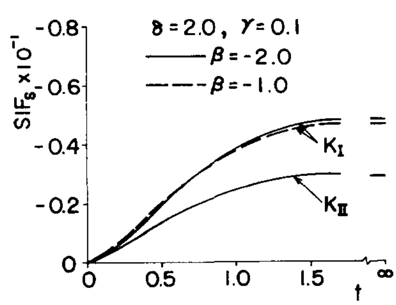

Fig. 6. The variation of SIFs with time t for different β, δ and γ.

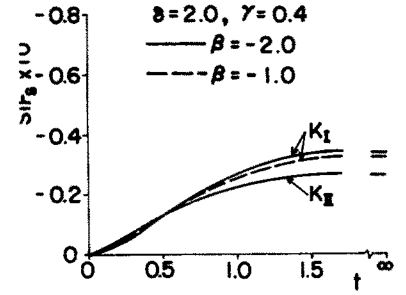

Fig. 7. The variation of SIFs with time t for different β, δ and γ.

different values of the mechanical nonhomogeneous parameter β and thermal nonhomogeneous parameters δ and γ. The nonhomogeneous parameter ε, as stated before, doesn't affect SIFs. Since K_{II} doesn't vary with β dramatically, only the results for $\beta = -2$ are given here for K_{II}. The following facts can be found from these figures. Firstly, SIFs increase with the time from their initial zero value to the maximum values at the steady state in the range of β, δ and γ considered in this paper. SIFs reach the steady state values at about $t = 1.5$ to 2.0. Secondly, K_{II} can be reduced substantially by selecting β, δ and γ appropriately. The maximum absolute value of K_{II} for $(\beta, \delta, \gamma) = (-2.0, 2.0, 0.1)$ is only about 40% of that for the homogeneous medium (Tsuji *et al.*, 1986). Finally, K_I is negative for the nonhomogeneous material in the range of β, δ and γ considered in this paper while it is positive for the homogeneous medium (Tsuji *et al.*, 1986). In fact, there exist values of β, δ and γ at which K_I becomes zero and when β, δ and γ are varied in an appropriately selected region, K_I becomes negative so that the contact of the crack faces would occur. The results presented here without considering this effect may not be exactly correct but would be more conservative, since the contact of the crack faces will increase the friction between the faces and make heat transfer across the crack faces easier. Thus the stress intensity factors would be lowered by these two factors. Recalling that one of the objectives of developing FGM materials is to reduce thermal stresses and thermal stress intensity factors, the applicability of the present solution may not be seriously affected without considering the contact of crack faces. For a more clear comparison, the SIFs for the homogeneous medium (β, δ, γ and ε all are zero) vs the dimensionless time t are depicted in Fig. 8 for $a = 1$ and 2. The results agree with those by Tsuji *et al.* (1986). And the SIFs at $(\beta, \delta, \gamma) = (-2.0, 2.0, 0.1)$ for $a = 1$ and 2 are shown in Fig. 9. In fracture mechanics, we know that the maximum of the cleavage stress σ_θ near the crack tip can roughly characterize the nature of crack initiation under mixed mode fracture. In Fig. 10, the maximum of the cleavage stress σ_θ for both homogeneous medium and $(\beta, \delta, \gamma) = (-2.0, 2.0, 0.1)$ are depicted for $a = 1$ and 2. It is clear that the reduction in the maximum value is significant. From these results, some suggestions on fabrication of FGM materials may

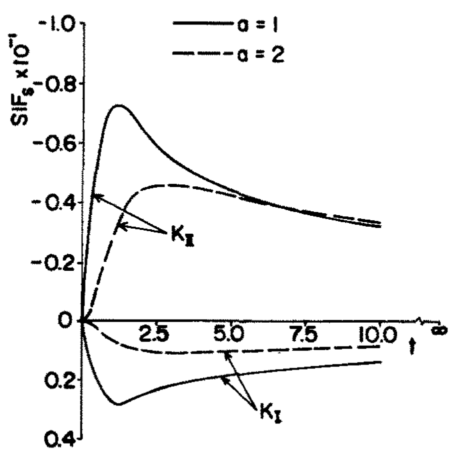

Fig. 8. The variation of SIFs with time t for the homogeneous medium.

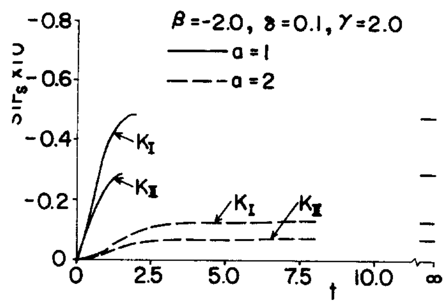

Fig. 9. The variation of SIFs with time t for $(\beta, \delta, \gamma) = (-2.0, 2.0, 0.1)$.

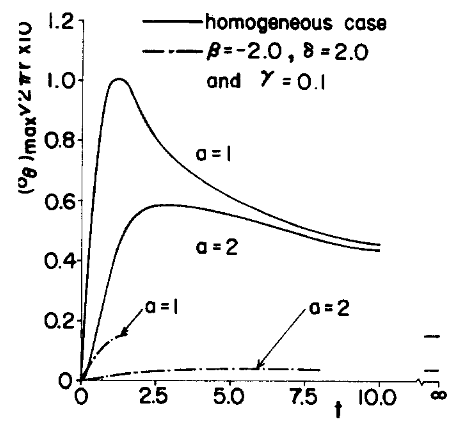

Fig. 10. The variation of the maximum cleavage stress σ_θ with time t for the homogeneous medium and $(\beta, \delta, \gamma) = (-2.0, 2.0, 0.1)$.

be made. Firstly, the elastic modulus of the ceramic (exposed to high temperature) in FGM should be higher than that of the metal ($\beta < 0$). Secondly, the heat conductivity and the thermal expansion coefficient of the ceramic should be lower than that of the metal ($\delta > 0$ and $\gamma > 0$). And finally, the variation in thermal expansion coefficient from the ceramic to the metal should be less dramatic than that in elastic modulus and heat conductivity ($\gamma < \delta$ and $\gamma < -\beta$).

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APPENDIX A. FUNCTIONS USED IN SECTION 4

$$\begin{aligned} h_{11}(\xi) &= -s_1+\exp[(s_1-s_2)a][s_1+as_2(s_2-s_1)] \\ h_{12}(\xi) &= 1-\exp[(s_1-s_2)a][1-a(s_2-s_1)(1-as_2)] \\ h_{21}(\xi) &= 1-\exp[(s_1-s_2)a][1+a(s_2-s_1)] \\ h_{22}(\xi) &= a^2(s_2-s_1)\exp[(s_1-s_2)a], \end{aligned} \tag{A1}$$

$$\begin{aligned} f_{11}(\xi) &= [-\beta h_{11}+s_2(s_1-s_2)h_{12}](s_1-s_2)^{-3} \\ f_{12}(\xi) &= [-2\xi h_{11}-\xi(s_1-s_2)h_{12}](s_1-s_2)^{-3} \\ f_{21}(\xi) &= [-\beta h_{21}+s_2(s_1-s_2)h_{22}](s_1-s_2)^{-3} \\ f_{22}(\xi) &= [-2\xi h_{21}-\xi(s_1-s_2)h_{22}](s_1-s_2)^{-3}, \end{aligned} \tag{A2}$$

$$\begin{aligned} g_1(\xi,p) &= -s_2^2 f_3-2s_2 f_4+f_5 \\ g_2(\xi,p) &= -(2s_2+\beta)s_2^2 f_3-(3s_2+2\beta)s_2 f_4-f_6 \\ g_3(\xi,p) &= \exp(-s_2 a)[(1-as_2)f_2-as_2^2 f_1]-(\beta+\gamma-\mu_1)C_{21}-(\beta+\gamma-\mu_2)C_{22} \\ g_4(\xi,p) &= \exp(-s_2 a)[(1+as_2)f_1+af_2]-C_{21}-C_{22}, \end{aligned} \tag{A3}$$

$$\begin{aligned} f_1(\xi,p) &= C_{21}\exp[-(\beta+\gamma-\mu_1)a]+C_{22}\exp[-(\beta+\gamma-\mu_2)a] \\ f_2(\xi,p) &= (\beta+\gamma-\mu_1)C_{21}\exp[-(\beta+\gamma-\mu_1)a]+(\beta+\gamma-\mu_2)C_{22}\exp[-(\beta+\gamma-\mu_2)a] \\ f_3(\xi,p) &= C_{12}-C_{21}-C_{22} \\ f_4(\xi,p) &= -(\beta+\gamma-\mu_1)C_{21}-(\beta+\gamma-\mu_2)(C_{22}-C_{12}) \\ f_5(\xi,p) &= (\beta+\gamma-\mu_1)^2C_{21}+(\beta+\gamma-\mu_2)^2(C_{22}-C_{12})+D(\xi,p) \\ f_6(\xi,p) &= (\beta+\gamma-\mu_1)^2(\gamma-\mu_1)C_{21}+(\beta+\gamma-\mu_2)^2(\gamma-\mu_2)(C_{22}-C_{12})+\gamma D(\xi,p). \end{aligned} \tag{A4}$$

In these functions, $C_{12}(\xi, p)$, $C_{21}(\xi, p)$ and $C_{22}(\xi, p)$ are given by (34) and $D(\xi, p)$ is given by (26).

APPENDIX B. NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

Assume that the Laplace transform of any function $F(t)$ is known as:

$$F^*(p) = \int_0^\infty F(t)\exp(-pt)\,dt. \tag{B1}$$

To begin with, the following variable substitutions are made:

$$\tau = 2\exp(-\eta t)-1, \tag{B2}$$

and

$$F(t) = F\left[-\frac{1}{\eta}\ln\left(\frac{1+\tau}{2}\right)\right] = W(\tau). \tag{B3}$$

Thus the eqn (B1) takes the form:

$$F^*(p) = \frac{1}{2\eta}\int_{-1}^{1}\left(\frac{1+\tau}{2}\right)^{p/\eta^{-1}} W(\tau)\,\mathrm{d}\tau. \tag{B4}$$

The function $W(t)$ is now expanded into a series of the Jacobi polynomials:

$$W(\tau) = \sum_{n=0}^{\infty} c_n P_n^{(0,\omega)}(\tau), \tag{B5}$$

where the Jacobi polynomials are defined as:

$$P_n^{(0,\omega)}(\tau) = \frac{(-1)^n}{2^n n!}(1+\tau)^{-\omega}\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}[(1-\tau)^n(1+\tau)^{\omega+n}], \tag{B6}$$

which form an orthogonal complete set of functions on the interval $[-1, 1]$. Now $F^*(p)$ is evaluated at the discrete points along the real positive p-axis in the Laplace transform plane given by:

$$p_i = (\omega+1+i)\eta, \qquad i = 0, 1, 2, \ldots, \tag{B7}$$

where $\omega > -1$ and $\eta > 0$ are real parameters.

By substituting (B5) into (B4) and using the orthogonality of Jacobi polynomials, the following set of equations is obtained to determine the coefficients c_n:

$$\sum_{n=0}^{k}\frac{k(k-1)\ldots[k-(n-1)]}{(k+\omega+1)(k+\omega+2)\ldots(k+\omega+1+n)}c_n = \eta F^*[(\omega+1+k)\eta], \qquad k = 1, 2, \ldots \tag{B8a}$$

$$\frac{c_0}{\omega+1} = \eta F^*[(\omega+1)\eta], \qquad k = 0. \tag{B8b}$$

Once a finite number of N coefficients c_n are calculated out, the function $F(t)$ can be approximately evaluated by the following formula:

$$F(t) = W(\tau) \approx \sum_{n=0}^{N} c_n P_n^{(0,\omega)}(\tau) = \sum_{n=0}^{N} c_n P_n^{(0,\omega)}[2\exp(-\eta t)-1]. \tag{B9}$$

The parameters ω, η and N are chosen such that $F(t)$ can be best described.